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PERIODIC SOLUTIONS OF $x'' + g(x) + \mu h(x) = 0$

BY

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ABSTRACT. Necessary and sufficient conditions for $x'' + f(x) = 0$ to admit at least one nontrivial periodic solution are given. The results are applied to $x'' + g(x) + \mu h(x) = 0$, $x(0) = A$, $x'(0) = 0$ in order to characterize those regions of the (μ, A)-plane for which nontrivial periodic solutions exist. A converse theorem is given, together with some illustrative examples.

1. Introduction. The problem of determining the existence of nontrivial periodic solutions to the equation

$$x'' + f(x) = \epsilon p(t) \tag{1}$$

has received considerable attention. In order to investigate this problem, it is necessary to consider the problem of determining periodic solutions of the corresponding unforced equation

$$x'' + f(x) = 0. \tag{2}$$

The existence of periodic solutions of (2) has been studied, for example, by Loud [7], Opial [12], Cesari [2] and Utz [16].

Recently, Maekawa [9] has considered the construction of periodic solutions of the equation

$$x'' + x + \mu x^2 = \epsilon \cos wt, \tag{3}$$

with the initial condition $x(0) = A$ (> 0), $x'(0) = 0$.

In the case that $\epsilon = 0$, he determined that periodic solutions exist if $0 < \mu A < ½$.

In the following section, we shall obtain necessary and sufficient conditions for the existence of periodic solutions of equation (2) under rather general

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conditions on f. In §3 we shall apply these results to characterize those points of the (μ, A)-plane for which the initial value problem

(4) $$x'' + g(x) + \mu h(x) = 0,$$

$x(0) = A$, $x'(0) = 0$, $\mu > 0$ has at least one nontrivial periodic solution.

In §4, we state and prove a converse theorem, and in §5 several examples illustrating our results will be given.

Throughout this paper, the following notation will be used:

$$F(y) \equiv \int_0^y f(u)\,du, \qquad G(y) \equiv \int_0^y g(u)\,du, \qquad H(y) \equiv \int_0^y h(u)\,du.$$

(5) $$\phi(\mu, y) \equiv G(y) + \mu H(y).$$

Further, in what follows, by periodic solution we shall always mean a nontrivial (i.e. nonconstant) periodic solution.

2. **Periodic solutions of** $x'' + f(x) = 0$. The following theorem generalizes known results (see, for example, [2]).

Theorem 1. *Let $f(x)$ be locally integrable on $(-\infty, \infty)$. Then a necessary and sufficient condition for there to be a periodic solution of* (2) *is that there exist real numbers α, β with $\alpha < \beta$ such that $F(\alpha) = F(\beta) > F(x)$ for $\alpha < x < \beta$.*

Proof of necessity. Assume that $z(t)$ is a periodic solution of (2) with period ω. The energy equation satisfied by $z(t)$ is

(6) $$\tfrac{1}{2} z'^2(t) + F(z(t)) = \text{const} = E, \text{ say.}$$

Let $m = \min_{0 \le t \le \omega} z(t)$, $M = \max_{0 \le t \le \omega} z(t)$. Then $m < M$. Let $t_1 = \inf\{t: t \ge 0$ and $z(t) = m\}$ and let $t_2 = \inf\{t: t \ge t_1$ and $z(t) = M\}$. We have $F(m) = F(M) = E \ge F(x)$ for $m < x < M$. Choose t_0 with $t_1 < t_0 < t_2$ such that $z'(t_0) > 0$. Let (t_3, t_4) be the maximal interval I about t_0 for which $z'(t) > 0$. Then $t_1 \le t_3 < t_0 < t_4 \le t_2$ and $z'(t_3) = z'(t_4) = 0$. Defining α to be $z(t_3)$, $\beta = z(t_4)$, the necessity follows.

Before proving the sufficiency, we require the following lemma, which is related to Sard's theorem [10], [11], [14], but does not appear to exist in the literature in the form in which we need it.

Lemma 1. *Let $F(x) \in AC[a, b]$. Let $C = \{y$: there exists $x \in [a, b]$ such that $y = F(x)$, $F'(x) = 0\}$ (the critical range of F). Then C has measure zero.*

Proof. Define $T = \{x: x \in [a, b]$ and $F'(x) = 0\}$. Let $\epsilon > 0$ be given. Since $F(x) \in AC[a, b]$, there exists $\delta > 0$ such that $\int_M |F'(x)|\,dx < \epsilon$ for any measurable set M with meas $M < \delta$. Let $I_1, I_2, \cdots, I_n$ be interior disjoint intervals such that $T \subset I = \bigcup_{j=1}^{n} I_j$ and meas $T > \Sigma_{j=1}^{n}$ (meas $I_j) - \delta$. Then $\int_I |F'(x)|\,dx = \int_{I \setminus T} |F'(x)|\,dx < \epsilon$ ('$\setminus$' denotes complement). But

$$\int_I |F'(x)|\,dx \geq \sum_{j=1}^{n} \left(\max_{x \in I_j} F(x) - \min_{x \in I_j} F(x) \right) = \sum_{j=1}^{n} \text{meas } D_j,$$

where $D_j = [\min_{x \in I_j} F(x), \max_{x \in I_j} F(x)]$. However $y \in C$ implies that $y = F(x)$ for some x belonging to some I_j, and hence $y \in D_j$ for some j. Thus $\bigcup_{j=1}^{n} D_j \supset C$, and meas $C < \epsilon$. Since $\epsilon > 0$ is arbitrary, the lemma follows.

Proof of sufficiency. Let $\alpha < \gamma < \beta$ such that $\Gamma = F(y) = \min_{[\alpha,\beta]} F(x)$. Let $E_0 = F(\alpha) = F(\beta)$ and for $\Gamma < y \leq E_0$, define $\xi(y)$ to be $\sup\{x\colon \alpha \leq x \leq y$ and $F(x) = y\}$ and $\eta(y)$ to be $\inf\{x\colon y \leq x \leq \beta$ and $F(x) = y\}$. Then $F(\xi(y)) = F(\eta(y)) > F(x)$ whenever $\xi(y) < x < \eta(y)$. By Lemma 1, there is a y^* with $\Gamma < y^* \leq E_0$ such that for every x for which $F(x) = y^*$, we have $F'(x) = f(x) \neq 0$. Define α^* to be $\xi(y^*)$ and β^* to be $\eta(y^*)$. Then, for $\alpha^* < x < \beta^*$, we have $F(\alpha^*) = F(\beta^*) > F(x)$, and, moreover,

$$F'(\alpha^*)F'(\beta^*) = f(\alpha^*)f(\beta^*) \neq 0. \tag{7}$$

Furthermore it is easily seen that $f(\alpha^*) < 0$, while $f(\beta^*) > 0$. We consider the initial value problem

$$x'' + f(x) = 0, \qquad x(0) = \alpha^*, \qquad x'(0) = 0. \tag{8}$$

Define $\phi(t)$ on its maximal domain $D \subset [0, \infty)$ by $\int_{\alpha^*}^{\phi(t)} du/\sqrt{E_0 - F(u)} = t$. (7) implies that there exists $\delta > 0$, $c > 0$ such that for $\alpha^* \leq u \leq \alpha^* + \delta$, we have $E_0 - F(u) \geq c(u - \alpha^*)$, and for $\beta^* - \delta \leq u \leq \beta^*$, we have $E_0 - F(u) \geq c(\beta^* - u)$. For $\alpha^* + \delta \leq u \leq \beta^* - \delta$, we have $E_0 - F(u) > c\delta$. It follows that

$$\int_{\alpha^*}^{\beta^*} \frac{du}{\sqrt{E_0 - F(u)}} \leq 4\frac{8}{c} + \frac{\beta^* - \alpha^*}{\sqrt{c\delta}}$$

which implies that $\phi(t)$ obtains the value β^* for $t = \frac{1}{2}T$, where $T > 0$. Defining $\phi(t)$ to be $\phi(T - t)$, $\frac{1}{2}T < t \leq T$, and then extending it periodically we obtain a periodic solution of (2). This completes the proof of the theorem.

Remark. Had we assumed the existence of solutions of IVP's for (2), the existence of a periodic solution would have followed from (7) and classical considerations. The following corollary will prove useful in the sequel.

Corollary 1. *Let $f(x)$ be continuous and assume uniqueness of IVP's for (2). Let $f(\alpha) < 0$ ($f(\alpha) > 0$). Then a necessary and sufficient condition for the solution of $x'' + f(x) = 0$, $x(0) = \alpha$, $x'(0) = 0$ to be periodic is that there exist $\beta > \alpha$ ($\beta < \alpha$) such that $F(\alpha) = F(\beta) > F(y)$ for $\alpha < y < \beta$ ($\beta < y < \alpha$).*

3. Admissible regions for equation (4). Now we shall concern ourselves with those regions of the $\mu - A$ plane, $\mu > 0$, for which equation (4) has a periodic solution.

Definition 1. The pair (μ, A) is said to be admissible ($(\mu, A) \in \mathcal{A}$) if equation (4) has a periodic solution.

In the rest of this paper we shall assume the following: (i) $g(x)$, $h(x)$ are continuous for all x; (ii) solutions of equation (4) are unique.

Theorem 2. *$\mathcal{A}$ is open.*

Proof. Let $(\mu_0, A_0) \in \mathcal{A}$. By Corollary 1, there exists B_0 such that $\phi(\mu_0, A_0) = \phi(\mu_0, B_0) > \phi(\mu_0, y)$, for y between A_0 and B_0. Assume, without loss of generality, that $B_0 < A_0$. Then $\partial\phi(\mu_0, B_0)/\partial y < 0$ and $\partial\phi(\mu_0, A_0)/\partial y > 0$. Hence there exist $C_1, C_2, \delta_1, \delta_2 > 0$ such that if $|\mu - \mu_0| < \delta_1$, $|B - B_0| < \delta_1$, then $\partial\phi(\mu, B)/\partial y < -C_1 < 0$, and $\partial\phi(\mu, y)/\partial\mu < C_2$, provided $B_0 - \delta_2 \le y \le (A_0 + B_0)/2$. Hence, if $|\mu - \mu_0| < \delta_1$, $|B - B_0| < \delta_2 \le \delta_1$, and $B_0 - \frac{1}{2}\delta_2 \le y \le \frac{1}{2}(A_0 + B_0)$, then $\phi(\mu, B) > \phi(\mu, y)$, for $B < y$. A similar argument shows that there exists a $\delta_3 > 0$ such that if $|\mu - \mu_0| < \delta_3$, $|A - A_0| < \delta_3$ and $\frac{1}{2}(A_0 + B_0) \le y < A$, then $\phi(\mu, A) > \phi(\mu, y)$. There exists $\delta_4 > 0$ such that $\delta_4 \le \min\{\delta_2, \delta_3, C_1\delta_2/4C_2\}$ for which $|\mu - \mu_0| < \delta_4$, $|A - A_0| < \delta_4$ implies that $|\phi(\mu, A) - \phi(\mu_0, A_0)| < C_1\delta_2/4$. For such μ, we have that $\phi(\mu, B_0 - \delta_2/2) - \phi(\mu_0, B_0) > \frac{1}{2}C_1\delta_2 - C_2\delta_4 > \frac{1}{4}C_1\delta_2$, since in this range $\partial\phi(\mu, B)/\partial y < -C_1$ and $|\partial\phi(\mu, y)/\partial\mu| < C_2$. Similarly $\phi(\mu_0, B_0) - \phi(\mu, B_0 + \frac{1}{2}\delta_2) > \frac{1}{4}C_1\delta_2$. Hence there exists B with $|B - B_0| < \frac{1}{2}\delta_2$ such that $\phi(\mu, A) = \phi(\mu, B)$. Further $\partial\phi(\mu, B)/\partial y < 0$, since $|B - B_0| < \delta_2$ and $|\mu - \mu_0| < \delta_1$. The theorem is now proved.

Definition 2. Let $(\mu_0, A_0) \in \partial\mathcal{A}$ such that $\mu_0 > 0$. Then we shall say that (μ_0, A_0) is of type I if $\partial\phi(\mu_0, A_0)/\partial y = 0$; is of type II if it is in $\mathrm{cl}\{(\mu, A) \in \partial\mathcal{A}$: there exists $B \neq A$ such that $\phi(\mu, B) = \phi(\mu, A)$ and $\partial\phi(\mu, B)/\partial y = 0$ and (μ, A) is not of type I$\}$; and is of type III if it is in $\mathrm{cl}\{(\mu, A) \in \partial\mathcal{A}$: $\phi(\mu, A) > \phi(\mu, y)$, either for all $y > A$ or for all $y < A$, and (μ, A) is not of type I or II$\}$.

Theorem 3. *Let $(\mu_0, A_0) \in \partial\mathcal{A}$. Then one of the following is true*: $\mu = 0$, *or (μ, A) is of one of the types* I, II *or* III.

Proof. Suppose that none of the first three alternatives holds. Since $(\mu, A) \in \partial\mathcal{A}$, there is a sequence $(\mu_n, A_n) \in \mathcal{A}$ with $(\mu_n, A_n) \to (\mu, A)$. By the corollary to Theorem 1, there exist $B_n \neq A_n$ such $\phi(\mu_n, A_n) = \phi(\mu_n, B_n) > \phi(\mu_n, y)$ for $A_n < y < B_n$ (without loss of generality, we may assume that $A_n < B_n$). If a subsequence B_{n_k} converges to B, then $A \le B$. By continuity, $\phi(\mu, A) = \phi(\mu, B) \ge \phi(\mu, y)$ for $A \le y \le B$. Since $\partial\phi(\mu, A)/\partial y$ and $\partial\phi(\mu, B)/\partial y$ have opposite sign (otherwise (μ, A) would be of type I or II) we have $A < B$; if there exists C with $A < C < B$ and $\phi(\mu, A) = \phi(\mu, C)$, then we must have $\partial\phi(\mu, C)/\partial y = 0$ and again

(μ, A) is of type II. Hence $\phi(\mu, A) > \phi(\mu, y)$ for $A < y < B$. But this implies that $(\mu, A) \in \mathcal{A}$ which is a contradiction. It follows, therefore, that the sequence $B_n \to +\infty$, and so $\phi(\mu, A) > \phi(\mu, y)$ for $y > A$ (equality is again excluded as (μ, A) would be of type II). Thus (μ, A) is of type III, and the theorem is proved.

It should be noted that a point of ∂A may be simultaneously more than one of the types I, II or III.

We now proceed to investigate the nature of the boundary of $\mathcal{A}$. In the next several theorems we show that under suitable hypotheses, boundary points exclusively of a given type (I, II or III) are interior to a continuous arc Γ of such points. Suppose that $xg(x) > 0$ for $x \neq 0$. The existence of periodic solutions in the case $\mu = 0$ has been much discussed in the literature (see, for example, [2] – [4], [6], [7], [9], [12]). Clearly nothing additional is obtained when $\mu > 0$, $xh(x) > 0$ for $x \neq 0$. Thus the case $h(x) > 0$ (or $h(x) < 0$) for all x will be of some interest. For this case we show that the above mentioned arc is *strictly decreasing*, where by strictly decreasing we mean $\Gamma = \{(\mu(s), A(s)) : s_0 \leq s < s_1\}$ with $\mu(s)$ monotone increasing and $A(s)$ strictly decreasing.

Theorem 4. (a) *Let (μ_0, A_0) be a point of type* I *exclusively such that $h(A_0) \neq 0$. Then it is relatively interior to a continuous arc of such points.*

(b) *Let (μ_0, A_0) be a point of type* II *exclusively, such that $g(x)$, $h(x)$ are continuously differentiable in some neighbourhood of B_0 (see Definition* 2) *and such that*

$$h(B_0)[H(B_0) - H(A_0)] \frac{d}{dB}\left(\frac{g(B)}{h(B)}\right)\Bigg|_{B=B_0} \neq 0.$$

Then (μ_0, A_0) is relatively interior to a continuous arc of points of type II.

(c) *Let (μ_0, A_0) be of type* III *exclusively, and suppose that $\phi(\mu_0, A_0) > \phi(\mu_0, y)$ for all $y > A_0$. Assume that $\sup_{0 \leq y < \infty} H(y) = \hat{H}$, and $\sup_{0 \leq y < \infty} G(y) = \hat{G}$ are finite, and that $\sup_{0 \leq x < \infty} \phi(\mu, x) = \hat{G} + \mu\hat{H}$ in some neighbourhood of μ_0. Then (μ_0, A_0) is relatively interior to a continuous arc of points of type* III. *A corresponding result holds if $\phi(\mu_0, A_0) > \phi(\mu_0, y)$ for all $y < A_0$.*

Proof. (a) (μ_0, A_0) is of type I implies that $g(A_0) + \mu_0 h(A_0) = 0$. Since $h(A_0) \neq 0$, we have, by the continuity of h, that there is a neighbourhood of A_0 in which $h(A) \neq 0$. In this neighbourhood define $\mu(A)$ to be $-g(A)/h(A)$. This defines a continuous arc Γ containing (μ_0, A_0) in its relative interior. We note that $\Gamma \cap \mathcal{A} = \emptyset$. Let $N_{\epsilon,\delta}$ be the rectangular neighbourhood $\{(\mu, A): |\mu - \mu_0| < \epsilon, |A - A_0| < \delta\}$. Since $(\mu_0, A_0) \in \partial\mathcal{A}$, $N_{\epsilon,\delta} \cap \mathcal{A} \neq \emptyset$; also $N_{\epsilon,\delta} \cap \mathcal{A}^c$ contains points other than (μ_0, A_0). It follows that $N_{\epsilon,\delta} \cap \partial\mathcal{A}$ contains points other than (μ_0, A_0). Since (μ_0, A_0) is a point exclusively of type I, it follows from considerations of continuity that we may choose δ, ϵ so small that $\partial\mathcal{A} \cap N_{\epsilon,\delta}$ consists of points exclusively of type I. Since all points of type I satisfy $g(A) + \mu h(A) = 0$, it follows that $\partial\mathcal{R} \cap N_{\epsilon,\delta} \subset \Gamma$ (here we use $h(A) \neq 0$). Since Γ is in fact the graph of a continuous function, it is clear that we may, in addition, choose δ, ϵ so small that $N_{\epsilon,\delta} \cap \Gamma$ is a continuous arc (the graph of a continuous function) containing (μ_0, A_0) in its relative interior. If $(\mu_1, A_1) \in N_{\epsilon,\delta} \cap \Gamma$, $(\mu_2, A_2) \in N'_{\epsilon,\delta} \cap \mathcal{R}$, we may join (μ_1, A_1), (μ_2, A_2) by a continuous arc intersecting Γ only at (μ_1, A_1). Since this arc must intersect $\partial\mathcal{R}$, it follows that $(\mu_1, A_1) \in \partial\mathcal{R}$. Thus $\partial\mathcal{R} \cap N_{\epsilon,\delta} = N_{\epsilon,\delta} \cap \Gamma$ and the proof of (a) is complete.

(b) We have $g(B_0) + \mu_0 h(B_0) = 0$ and $\phi(\mu_0, A_0) = \phi(\mu_0, B_0)$. Define $J(\mu, A, B)$ to be $\phi(\mu, B) - \phi(\mu, A)$. Since $h(B_0) \neq 0$, there exists $\delta > 0$ such that $h(B) \neq 0$ for $|B - B_0| < \delta$. For B in this interval, define $\mu(B)$ to be $-g(B)/h(B)$. Note that $\mu(B_0) = \mu_0$. Consider now $J(\mu(B), A, B)$.

$$J(\mu(B_0), A_0, B_0) = \phi(\mu_0, B_0) - \phi(\mu_0, A_0) = 0,$$

$$\frac{\partial J}{\partial B}(\mu(B), A, B) = \frac{\partial J}{\partial \mu}(\mu, A, B)\frac{d\mu(B)}{dB} + \frac{\partial J}{\partial B}(\mu, A, B)$$

$$= [H(B) - H(A)]\frac{d}{dB}\left(\frac{g(B)}{h(B)}\right) + g(B) + \mu(B)h(B).$$

Thus

$$\frac{\partial J}{\partial B}(\mu(B_0), A_0, B_0) = [H(B_0) - H(A_0)]\frac{d}{dB}\left(\frac{g(B_0)}{h(B_0)}\right) \neq 0,$$

by hypothesis. Hence, by the implicit function theorem, we can solve $J(\mu(B), A, B) = 0$ for $B = B(A)$ as a continuous function of A in a neighbourhood of A_0. Define $\mu(A)$ to be $\mu = (-g(B(A))/h(B(A)))$. Clearly, this defines a continuous arc of points containing (μ_0, A_0) in its relative interior. Furthermore, this is a unique such arc. The remainder of the proof follows as in part (a).

(c) First we show that our hypotheses imply that $\phi(\mu_0, A_0) = \sup_{A_0 \le y < \infty} \phi(\mu_0, y)$. For suppose $\phi(\mu_0, A_0) > \sup_{0 \le y < \infty} \phi(\mu_0, y)$. Define $\psi(\mu, A)$ to be $\phi(\mu, A) - \hat{G} - \mu\hat{H}$. Then $\psi(\mu_0, A_0) > 0$. Thus there exists a neighbourhood of (μ_0, A_0) in which $\psi(\mu, A) > 0$, i.e. $\phi(\mu, A) > \sup_{A \le y < \infty} \phi(\mu, y)$ in this neighbourhood. Hence there are no points of this neighbourhood in the set $\mathcal{R}$, contradicting the fact that $(\mu_0, A_0) \in \partial\mathcal{R}$.

It follows that $\phi(\mu_0, A_0) = \sup_{A_0 \le y < \infty} \phi(\mu_0, y)$, as stated.

Since $H(A_0) \neq \hat{H}$, choose $\delta > 0$ such that for $|A - A_0| < \delta$, we have $H(A) \neq \hat{H}$. Define $\mu(A)$ by $\mu(A) = -(G(A) - \hat{G})/(H(A) - \hat{H})$. Then $\phi(\mu, A) = G(A) + \mu H(A) = \hat{G} + \mu\hat{H} = \sup_{A \le y < \infty} \phi(\mu, y)$. Hence we have an arc of points containing (μ_0, A_0) in its relative interior. The remainder of the proof follows as in parts (a) and (b).

Remark 1. Some of the conditions required in the hypotheses of Theorem 4 are somewhat technical in nature, but they appear to be necessary for the method of proof adopted; for example in (a) if A_0 is an isolated zero of h, we may still define a continuous arc Γ by

$$\Gamma = \{(\mu, A) : \mu = -g(A)/h(A),\ A \neq A_0\} \cup \{(\mu_0, A_0)\};$$

however this may not be wholly contained in $\partial\mathcal{P}$.

Remark 2. The theorem may no longer hold if the assumption that (μ_0, A_0) is *exclusively* of type I, II or III is removed.

The case where $\mu = 0$, $xg(x) > 0$ for $x \neq 0$, and $A > 0$, has been extensively discussed. In the case that $\mu > 0$, $h(x) > 0$ when $x \neq 0$ is also of interest as was earlier remarked. In this case $(A > 0)$ (μ, A) cannot be a boundary point of type I. It is the purpose of the following theorem to obtain further information about points of type II.

Theorem 5. *Assume that* $xg(x) > 0$ *and* $h(x) > 0$ *for all* $x \neq 0$. *Let* (μ_0, A_0) *be a point of type* II *but not of type* III, *with* $\mu_0 A_0 > 0$. *Then there exists a continuous strictly monotone decreasing arc* $\Gamma = \{(\mu, A(\mu)) : \mu_0 \leq \mu < \mu^*\}$ *of such points, with* $\lim_{u \to \mu^*} A(\mu) = 0$ *if the maximal interval* $[\mu, \mu^*)$ *of definition of the arc is finite.*

Proof. Since (μ_0, A_0) is not of type III, there exists B_0 (< 0, by virtue of the hypotheses of the theorem) such that $\phi(\mu_0, B_0) = \phi(\mu_0, A_0)$. Choosing B_0 to be the largest value of B for which $\phi(\mu_0, B) = \phi(\mu_0, A_0)$, it follows that $\partial\phi(\mu_0, B_0)/\partial y = 0$, and $\phi(\mu_0, y) < \phi(\mu_0, A_0) = \phi(\mu_0, B_0)$ for $B_0 < y < A_0$.

Now $\phi(\mu, B_0) > 0$ for $\mu_0 \leq \mu < \mu^*$, say, and $\partial\phi(\mu, B_0)/\partial y = g(B_0) + \mu_0 h(B_0) + (\mu - \mu_0)h(B_0) > 0$ for $\mu > \mu_0$. Thus defining $B(\mu)$ to be (for $\mu_0 \leq \mu < \mu^*$) $\sup\{y : B_0 < y < 0$ and $\phi(\mu, y) = \sup_{B_0 < y < 0} \phi(\mu, B)\}$, we have $B_0 < B(\mu) < 0$ and $\partial\phi(\mu, B(\mu))/\partial y = 0$. Clearly $B(\mu_0) = B_0$.

We shall show that $B(\mu)$ is nondecreasing. Let $\mu_0 \leq \mu_1 < \mu_2 < \mu^*$ and let $B(\mu_i) = B_i$. Suppose $B_1 > B_2$. Then

$$\phi(\mu_2, B_2) = G(B_2) + \mu_2 H(B_2) > G(B_1) + \mu_2 H(B_1) = \phi(\mu_2, B_1),$$

i.e. $G(B_2) - G(B_1) > \mu_2[H(B_1) - H(B_2)]$. However,

$$g(B_2) + \mu_1 h(B_2) < g(B_2) + \mu_2 h(B_2) = 0$$

and so $\phi(\mu_1, B_2) < \sup_{B_0 \leq y \leq 0} \phi(\mu_1, y) = \phi(\mu_1, B_1)$. Therefore

$$0 < G(B_2) - G(B_1) < \mu_1[H(B_1) - H(B_2)] < \mu_2[H(B_1) - H(B_2)]$$

giving a contradiction. Thus $B_1 \leq B_2$, and $B(\mu)$ is nondecreasing.

Next we show that $\phi(\mu, B(\mu))$ is continuous in μ. Let $\mu_i \uparrow \mu$ with $B(\mu_i) \uparrow$

$B(\mu) - \epsilon$, where $0 \le \epsilon < \infty$. Then $\phi(\mu_i, B(\mu_i)) \to \phi(\mu, B(\mu) - \epsilon)$ and by continuity, we have $\phi(\mu, B(\mu) - \epsilon) \ge \phi(\mu, y)$ for $B(\mu) - \epsilon \le y \le 0$. It follows from the definition of $B(\mu)$ that ϵ must be 0, and so $\phi(\mu, B(\mu))$ is continuous to the left. Now let $\mu_i \downarrow \mu$ with $B(\mu_i) \downarrow B(\mu) + \epsilon$ $(0 \le \epsilon < \infty)$. We have $\phi(\mu_i, B(\mu_i)) \to \phi(\mu, B(\mu) + \epsilon)$. If $\epsilon = 0$, there is nothing to prove; otherwise we have $\phi(\mu, B(\mu)) > \phi(\mu, B(\mu) + \epsilon)$, on account of the definition of $B(\mu)$. It follows that for i sufficiently large, we have $\phi(\mu_i, B(\mu_i)) < \phi(\mu_i, B(\mu))$. Since $B_0 \le B(\mu) < B(\mu_i) < 0$, this clearly contradicts the definition of $B(\mu_i)$. It now follows that $B(\mu)$ is also continuous to the right and hence continuous. For $\mu_0 < \mu < \mu^*$, we have

$$\begin{aligned}\phi(\mu, A_0) &> \phi(\mu_0, A_0) = \phi(\mu_0, B_0) = G(B_0) + \mu_0 H(B_0)\\ &\doteq [G(B(\mu)) + \mu H(B(\mu))] + [(G(B_0) + \mu_0 H(B_0)) - (G(B(\mu)) + \mu_0 H(B(\mu)))]\\ &\qquad + (\mu_0 - \mu) H(B(\mu))\\ &> G(B(\mu)) + \mu H(B(\mu)) > \phi(\mu, 0) = 0,\end{aligned}$$

since $(\mu_0 - \mu)$ and $H(B(\mu))$ are negative, and the expression in the second square brackets is positive on account of the definition of B_0.

It follows that the equation $\phi(\mu, A) = \phi(\mu, B(\mu))$ has at least one solution A with $0 < A < A_0$; in fact, exactly one, since the hypotheses of the theorem imply that $\phi(\mu, A)$ is strictly increasing for $A > 0$. We shall denote this solution by $A(\mu)$.

Let $\mu_0 < \mu_1 < \mu^*$ and let $A_1 = A(\mu_1)$, $B_1 = B(\mu_1)$. Define η_1, η_2 by

$$\eta_1 = \min_{A_1/2 \le y \le 3A_1/2} g(y), \qquad \eta_2 = \max_{A_1/2 \le y \le 3A_1/2} h(y).$$

Let $\epsilon > 0$ and choose $\delta > 0$ so that $\delta < \epsilon\eta_1/2\eta_2$, and $\phi(\mu_1, B_1) - \phi(\mu, B(\mu)) < \frac{1}{2}\epsilon$ whenever $|\mu - \mu_1| < \delta$. Then if $|A - A_1| > \epsilon$ and $|\mu - \mu_1| < \delta$, we have

$$|\phi(\mu_1, A_1) - \phi(\mu, A(\mu))| \ge \epsilon\eta_1 - \delta\eta_2 > \tfrac{1}{2}\epsilon;$$

however $|\phi(\mu_1, B_1) - \phi(\mu, B(\mu))| < \frac{1}{2}\epsilon$ which is a contradiction since $\phi(\mu, A(\mu)) = \phi(\mu, B(\mu))$, and $\phi(\mu_1, A_1) = \phi(\mu_1, B_1)$. It follows that $A(\mu)$ is continuous for $\mu_0 < \mu < \mu^*$. Now let $\mu_0 \le \mu_1 < \mu_2 < \mu^*$, and let $A_i = A(\mu_i)$, $B_i = B(\mu_i)$, $i = 1, 2$. We have

$$\phi(\mu_2, B_2) = G(B_2) + \mu_2 H(B_2) < G(B_2) + \mu_1 H(B_2) \le G(B_1) + \mu_1 H(B_1) = \phi(\mu_1, B_1).$$

It follows that $\phi(\mu_2, A_2) < \phi(\mu_1, A_1)$ and so $A_2 < A_1$. Thus $A(\mu)$ is strictly decreasing in μ and $A(\mu)$ is defined in $[\mu_0, \mu^*)$. An examination of the definition of μ^* reveals that we require only that there exists for each $\mu \in [\mu_0, \mu^*]$, a B with $B_0 < B < 0$ such that $\phi(\mu, B(\mu)) > 0$. Defining μ^*, therefore, so that this interval is maximal, it follows that either $\mu^* = +\infty$, or $\mu^* < \infty$ and $\phi(\mu^*, B) \leq 0$ for $B_0 \leq B \leq 0$, and by continuity of ϕ and $A(\mu)$, we must have $\phi(\mu, A(\mu)) \to 0$ as $\mu \to \mu^*$. The hypotheses of the theorem will then imply that $\lim_{\mu \to \mu^*} A(\mu) = 0$. This completes the proof of the theorem.

Corollary 2. *The energy function $\phi(\mu, A)$ is decreasing along the arcs of type* II *defined in the above theorem.*

Proof. $\phi(\mu, A(\mu)) = \phi(\mu, B(\mu))$. So

$$\partial\phi(\mu, A(\mu))/\partial\mu = \partial\phi(\mu, B(\mu))/\partial\mu = [g(B(\mu)) + \mu h(B(\mu))]B'(\mu) + H(B(\mu))$$

for almost all μ, since $B(\mu)$ is monotone.

$$g(B(\mu)) + \mu h(B(\mu)) = 0 \quad \text{and so} \quad \partial\phi(\mu, A(\mu))/\partial\mu = H(B(\mu)) < 0.$$

4. A converse theorem. We now consider a converse problem, namely, given a decreasing continuous function $A(\mu)$, can functions $g(x)$, $h(x)$, $xg(x) > 0$, $h(x) > 0$ for $x \neq 0$ be found such that $A(\mu)$ is the boundary of the admissible set in the first quadrant, of $x'' + g(x) + \mu h(x) = 0$?

With some additional restrictions on $A(\mu)$, the next theorem shows that the answer is yes.

Theorem 6. *Let $A(\mu)$ be a positive, continuously differentiable function which is strictly decreasing for $\mu > 0$ such that $\lim_{\mu \to 0} A(\mu) = +\infty$ and $\lim_{\mu \to +\infty} A(\mu) = 0$. Then there exist $g(x)$, $h(x)$ with $g(-x) = -g(x)$ and $h(-x) = h(x) > 0$ for $x \neq 0$ such that*

$$x'' + g(x) + \mu h(x) = 0 \tag{9}$$

has $(\mu, A(\mu))$ as its only boundary points of $\mathfrak{A}$ in the interior of the first quadrant.

Proof. We wish to construct $g(x)$ and $h(x)$ with the above properties, such that $G(A) + \mu H(A) = G(B) + \mu H(B)$ and $g(B) + \mu h(B) = 0$, where $B = B(\mu) < 0$. Let there be a function $q(x)$, and constants $C_1 \neq 0$, C_2, such that

$$G(x) = C_1 q(x) g(x), \qquad H(x) = C_2 q(x) h(x). \tag{10}$$

Then $\mu H(B) = \mu C_2 q(B) h(B) = -C_2 q(B) g(B) = -C_2 G(B)/C_1$. We shall arrange that

$$\mu H(A) = C_3 G(A), \quad \text{for some constant } C_3. \tag{11}$$

From (10) and the definition of $G(x)$, $G(x) = C_1 q(x) G'(x)$. Solving gives

$$G(x) = G(x_0) \exp\left(C_1^{-1} \int_{x_0}^{x} q(s)^{-1}\, ds \right). \tag{12}$$

Similarly

(13) $$H(x) = H(x_0)\exp\left(C_2^{-1}\int_{x_0}^{x} q(s)^{-1}\,ds\right),$$

and hence

(14) $$H(x) = C_0G^{\alpha}(x), \quad \text{where } C_0 = H(x_0)/G^{\alpha}(x_0),\ \alpha = C_1/C_2.$$

In order for (11) to be valid we want $G(A) = C_3^{-1}\mu H(A) = \mu C_3^{-1}C_0G^{\alpha}(A)$. Hence $G^{\alpha-1}(A) = C_3C_0^{-1}\mu^{-1}$, or

(15) $$G(A) = (C_3C_0^{-1}\mu^{-1})^{1/(\alpha-1)}.$$

Define

(16) $$\psi(x) = \frac{1}{G(x_0)}\left(\frac{C_3}{C_0A^{-1}(x)}\right)^{1/(\alpha-1)},$$

where $A^{-1}(x)$ stands for the inverse function. Then

$$\psi(A(\mu)) = \frac{1}{G(x_0)}\left(\frac{C_3}{C_0\mu}\right)^{1/(\alpha-1)} = \frac{G(A)}{G(x_0)}$$

by (15). Hence, letting $G_0 = G(x_0)$, we define

(17) $$G(x) = G_0\psi(x) = \left(\frac{C_3}{C_0A^{-1}(x)}\right)^{1/(\alpha-1)}, \quad x \geq 0,$$

and then by (14)

(18) $$H(x) = C_0\left(\frac{C_3}{C_0A^{-1}(x)}\right)^{\alpha/(\alpha-1)}, \quad x \geq 0.$$

We extend the definition to all x by defining $G(-x) = G(x)$ and $H(-x) = -H(x)$. We observe that

(19) $$q(x) = -\frac{C_0}{C_3}\,\frac{(\alpha-1)}{\alpha}\left(\frac{C_3}{C_0A^{-1}(x)}\right)^{(\alpha-1)/\alpha}\frac{(A^{-1}(x))^2}{(A^{-1}(x))'}.$$

By the hypotheses on $A(x)$, $g(x)$ and $h(x)$ are well defined and continuous, and are such that $xg(x) > 0$, $x \neq 0$ and $h(x) > 0$, $x \neq 0$.

It remains to ensure that

(20) $$G(A) + \mu H(A) = G(B) + \mu H(B) \quad \text{and} \quad g(B) + \mu h(B) = 0.$$

The first of the equations in (20) requires that

(21) $$(1 + C_3)(C_3/C_0\mu)^{1/(\alpha-1)} = (1 - C_2/C_1)(-C_2/C_0C_1\mu)^{1/(\alpha-1)},$$

the last factor on the right being obtained by combining $\mu H(B) = -C_2G(B)/C_1$ with (14). Equation (21) reduces after simplification to

(22) $$C_3^{1/(\alpha-1)}(1 + C_3) = (1 - 1/\alpha)(1/\alpha)^{1/(\alpha-1)},$$

which is a constraint on C_1, C_2, C_3. C_0 may be chosen arbitrarily. The second of equations (20) is satisfied automatically.

Since, by the constructed properties of $g(x)$ and $h(x)$, for μ, $A > 0$ there can occur boundary points only of type II, the theorem is proved.

5. **Examples.** We first give an example to show that the boundaries of $\mathcal{R}$ can be very complicated.

Example 1. Let $\mathcal{C}$ be the obvious extension to the real line of the Cantor set (removing middle thirds) on [0, 1]. Let

(23) $$g(x) = x, \qquad h(x) = \begin{cases} 0, & x \geq 0, \\ (\rho(x, \mathcal{C}) - 1)x, & x < 0, \end{cases}$$

where $\rho(x, \mathcal{C}) = \inf_{y \in \mathcal{C}} |x - y|$. Note $0 \leq \rho(x, \mathcal{C}) \leq 1/6$. Then

(24) $$g(x) + \mu h(x) = \begin{cases} x, & x \geq 0, \\ x + \mu x(\rho - 1), & x < 0. \end{cases}$$

If $\mu < 1$, $\phi(\mu, x) > 0$, $x \neq 0$ and $\lim_{x \to -\infty} \phi(\mu, x) = +\infty$, and $g(x) + \mu h(x) \neq 0$ for $x \neq 0$ and hence all solutions are periodic.

For $\mu = 1$, $g(x) + \mu h(x) = 0$ for $x < 0$ and $x \in \mathcal{C}$. $\phi(\mu, x)$ is monotone decreasing for $x < 0$.

For $1 < \mu < 6/5$, there will be continuous boundary curves (of types I and II for $A < 0$ and of type II for $A > 0$) emanating from the Cantor set of boundary points for $\mu = 1$ and they must decrease in the case $A > 0$ to the μ-axis between $1 < \mu < 6/5$. For $\mu > 6/5$ there are no periodic solutions.

Note that using a Cantor set of positive linear measure, it is possible to obtain a boundary set of positive 2-dimensional measure.

Example 2. Consider the equation

(25) $$x'' + x + \mu \sum_{i=1}^{n} C_i x^{2i} = 0, \qquad x(0) = A > 0,\ x'(0) = 0.$$

Theorem 7. *Let*

(26) $$\theta(\mu, A) = 2\left[\sum_{i=1}^{n} (\mu C_i)^{2/(2i-1)}\right]\phi(\mu, A).$$

If $C_1 > 0$, $C_i \geq 0$ $(i = 2, \cdots, n-1)$, $C_n > 0$, *then* $\{(\mu, A) | \mu > 0, A > 0, \theta(\mu, A) < 1/3\} \subset \mathfrak{A}$. *Further if* $\alpha > 1/3$, *there exists* $(\mu_0, A_0) \in R^2 \backslash \mathfrak{A}$ *such that* $\theta(\mu_0, A_0) = \alpha$.

We first need the following lemma.

Lemma 2. *Let*

$$\tilde{\theta}(w) = \left(\sum_{i=1}^{n} w_i^{2/(2i-1)}\right)\left(1 - \sum_{i=1}^{n} \frac{2}{2i+1} w_i\right),$$

where $w = (w_1, w_2, \cdots, w_n)$. *If* $w_i \geq 0$ *and* $\Sigma_{i=1}^n w_i = 1$, *then* $\min \tilde{\theta}(w) = 1/3$.

Proof. If $n = 1$, then $\tilde{\theta}(w) = w_1^2(1 - 2/3)w_1$. $w_1 = 1$ implies that $\tilde{\theta}(w) = 1/3$. If $n = 2$, $w_1 + w_2 = 1$, then

$$\tilde{\theta}(w) - \frac{1}{3} = (w_1^2 + w_2^{2/3})\left(1 - \frac{2}{3}w_1 - \frac{2}{5}w_2\right) - \frac{1}{3} = [(1 - w_2)^2 + w_2^{2/3}]\left[\frac{1}{3} + \frac{4}{15}w_2\right] - \frac{1}{3}$$

$$= [(1 - w_2)^2 + w_2^{2/3}]\frac{4}{15}w_2 - \frac{1}{3}[1 - (1 - w_2)^2 - w_2^{2/3}]$$

$$= \frac{1}{15}[5w_2^{2/3} - 6w_2 + 4w_2^{5/3} - 3w_2^2 + 4w_2^3] \geq \frac{1}{15}[5w_2^{2/3} - 6w_2 + 4w_2^{5/3} - 3w_2^2]$$

$$= \frac{1}{15}[w_2^{2/3}(1 - w_2^{1/3})(5 - w_2^{1/3} - w_2^{2/3}) + 3w_2^{5/3}(1 - w_2^{1/3})] \geq 0$$

since $0 \leq w_2 \leq 1$, and $\tilde{\theta}(w) = 0$ for $w_2 = 0$. Hence the lemma is true for $n = 2$.

Suppose now the lemma is true for $1 \leq n \leq N$. Let $w_i \geq 0$, $i = 1, \cdots, N+1$, $\Sigma_{i=1}^{N+1} w_i = 1$. Let $w_N + w_{N+1} = v_N$. Then

$$\left(1 - \frac{2}{3}w_1 - \frac{2}{5}w_2 - \cdots - \frac{2}{2N+3}w_{N+1}\right)$$

$$\geq \left(1 - \frac{2}{3}w_1 - \cdots - \frac{2}{2N-1}w_{N-1} - \frac{2}{2N+1}v_N\right)$$

and

$$w_1^2 + w_2^{2/3} + \cdots + w_{N+1}^{2/(2N+1)} \geq w_1^2 + \cdots + w_{N-1}^{2N-3} + v_N^{2/(2N-1)}$$

since

$$(w_N + w_{N+1})^{2/(2N-1)} \leq w_N^{2/(2N-1)} + w_{N+1}^{2/(2N-1)} \leq w_N^{2/(2N-1)} + w_{N+1}^{2/(2N+1)}$$

for $0 \leq w_{N+1} \leq 1$.

This is equivalent to the case $n = N$ and hence the lemma is proved by induction, since in each case $\tilde{\theta}(w) = 1/3$ for $w_1 = 1$ and $w_i = 0$, $i > 1$.

Proof of Theorem 7. By [7] we know that $(\mu, A) \in \mathfrak{R}$ if

$$x\left(x + \mu \sum_{i=1}^{n} C_i x^{2n}\right) > 0, \quad B \le x \le A, \tag{27}$$

where $B < 0$ is given by $\phi(\mu, B) = \phi(\mu, A)$. From (27) we have that

$$\sum_{i=1}^{n} \mu C_i (-B)^{2i-1} < 1 \tag{28}$$

and also that

$$\theta(\mu, A) = \theta(\mu, B). \tag{29}$$

Let $w_i = \mu C_i(-x)^{2i-1}$, $i = 1, 2, \cdots, n$. Then

$$\theta(\mu, x) = 2\left[\sum_{i=1}^{n} w_i^{2/(2i-1)} \cdot \frac{1}{x^2}\right]\left[\frac{x^2}{2} - \sum_{i=1}^{n} \frac{1}{2i+1} x^2 w_i\right]$$

$$= \left[\sum_{i=1}^{n} w_i^{2/(2i-1)}\right]\left[1 - \sum_{i=1}^{n} \frac{2w_i}{2i+1}\right] = \tilde{\theta}(w).$$

Further, the constraint (28) becomes, for $x = B$,

$$\sum_{i=1}^{n} w_i < 1. \tag{30}$$

Let $0 < \theta_0 < 1/3$ be given. Let $\mu > 0$ and $A > 0$ be such that $\theta(\mu, A) = \theta_0$. Define Γ to be the manifold

$$\frac{w_1}{\mu C_1} = \left(\frac{w_2}{\mu C_2}\right)^{1/3} = \cdots = \left(\frac{w_n}{\mu C_n}\right)^{1/(2n-1)}, \quad w_i \ge 0 \tag{31}$$

(eliminating those terms for which $C_i = 0$). This is a connected manifold which intersects the origin $(\tilde{\theta}(0) = 0)$ and the boundary $\Sigma_{i=1}^{n} \underset{\sim}{w}_i = 1$ $(\tilde{\theta} \ge 1/3)$. Then by continuity, there exists $w_1^*, \cdots, w_n^*$ on Γ such that $\tilde{\theta}(w^*) = \theta_0$. Hence there exists $B < 0$ such that $\theta(\mu, B) = \theta_0$ with $B = (-w_i^*/\mu C_i)^{1/(2i-1)}$ and B satisfying (28).

Suppose now $\theta_0 > 1/3$. Let $(\mu_0, A_0) \in R_+^2$ with

$$\theta(\mu_0, A_0) = \theta_0 \quad \text{and} \quad \mu_0 > \max_{2 \le i \le n} \left\{1, \frac{n}{3\theta_0 - 1} \frac{C_i^{2/(2i-1)}}{C_i^2}\right\}.$$

By the above, for all w satisfying (30),

$$\frac{w_i^{2/(2i-1)}}{w_1^2} = \frac{(C_i \mu_0)^{2/(2i-1)}}{C_1^2 \mu_0^2} < \frac{C_i^{2/(2i-1)}}{C_1^2} \mu_0^{-1} < \frac{3\theta_0 - 1}{n}, \quad i = 2, \cdots, n.$$

Hence $\tilde{\theta}(w) < 3\theta_0 w_1^2(1 - 2w_1/3 - \cdots - 2w_n/(2n+1)) \leq 3\theta_0 w_1^2(1 - 2w_1/3) \leq \theta_0$. Hence for such μ_0, $B < 0$ having the required properties for periodic solutions does not exist. This proves the theorem.

Remark 3. A similar result holds for the equation $x'' + x + \mu \Sigma_{i=1}^n C_i x^{2(i+k)} = 0$, $x(0) = A > 0$, $x'(0) = 0$, $k \geq 0$, $C_i \geq 0$, C_1, $C_n > 0$. Let

$$\theta_k(\mu, x) = 2\left[\sum_{i=1}^{n} (\mu C_i)^{2/(2(k+i)-1)}\right]\phi(\mu, x).$$

Then the condition $\phi(\mu, A) < 1/3$ becomes $\theta_k(\mu, A) < 1 - 2/(2k+3)$.

Example 3. This is a special case of Example 2.

$$x'' + x + \mu x^{2n} = 0, \qquad x(0) = A > 0, \quad x'(0) = 0. \tag{32}$$

The condition $\theta_k(\mu; A) < 1 - 2/(2k+3)$ becomes

$$2\mu^{2/(2n-1)}(A^2/2 + \mu A^{2n+1}/(2n+1)) < 1 - 2/(2n+1). \tag{33}$$

This is satisfied if

$$\mu A^{2n-1} < \gamma_n, \tag{34}$$

where γ_n is a positive solution of

$$\gamma_n^2(1 + 2\gamma_n/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1}. \tag{35}$$

Suppose now that $\mu A^{2n-1} = \gamma_n$. For a periodic solution of equation (32) to exist, there must exist $B < 0$ such that

$$B^2\left(1 + \frac{2}{2n+1}\mu B^{2n-1}\right) = A^2\left(1 + \frac{2\mu A^{2n-1}}{2n+1}\right), \tag{36}$$

or

$$\begin{aligned}(\mu B^{2n-1})^2\left(1 + \frac{2}{2n+1}\mu B^{2n-1}\right)^{2n-1} &= \gamma_n^2\left(1 + \frac{2\gamma_n}{2n+1}\right)^{2n-1} \\ &= \left(\frac{2n-1}{2n+1}\right)^{2n-1} = \left(1 - \frac{2}{2n+1}\right)^{2n-1}.\end{aligned} \tag{37}$$

It is easily shown that the unique negative solution (up to multiplicities) of the equation

$$\gamma^2(1 + 2\gamma/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1} \tag{38}$$

is $\gamma = -1$. Hence $\mu B^{2n-1} = -1$, since $B < 0$. Hence $g(B) + \mu h(B) = B - B^{2n}/B^{2n-1} = 0$, and (μ, A) is on $\partial\mathfrak{A}$. This means that the curve

$$\mu A^{2n-1} = \gamma_n \tag{39}$$

is a boundary curve (of type II) for $\mathfrak{A}$.

Remark 4. It can be shown that γ_n is strictly decreasing and that $\lim_{n\to\infty} \gamma_n = \gamma$, where γ is the unique positive root of

$$x^2 e^{2x} = e^{-2}. \tag{40}$$

To prove this, we first consider $(1 - 2/(2n+1))^{2n-1}$. Let $y = (1 - 2/(x+2))^x$. Then

$$y'/y = \log(x/(x+2)) + 2/(x+2) = \log(1-\epsilon) + \epsilon < 0,$$

where $\epsilon = 2/(x+2)$, $x \geq 1$. Hence, since $y > 0$, $y' < 0$ and y is decreasing and so $((2n-1)/(2n+1))^{2n-1}$ is decreasing with n.

Now consider $(1 + 2\gamma_n/(2n+1))^{2n-1}$. Let $y = (1 + \delta/(x+2))^x$, where $\delta > 0$. Here

$$\frac{y'}{y} = \log\left(1 + \frac{\delta}{x+2}\right) - \frac{\delta x}{(x+2)(x+2+\delta)} > \frac{\delta}{x+2} - \frac{1}{2}\frac{\delta^2}{(x+2)^2} - \frac{\delta x}{(x+2)(x+\delta+2)}$$
$$= \frac{\delta}{x+2}\left(1 - \frac{1}{2}\frac{\delta}{x+2} - \frac{x}{x+\delta+2}\right) \geq \frac{\delta}{x+2}\left(1 - \frac{1}{2(x+2)} - \frac{x}{x+2}\right) = \frac{3\delta}{2(x+2)^2} > 0.$$

This means that $(1 + 2x/(2n+1))^{2n-1}$ is increasing with n for each fixed positive x.

Consider now the equation

$$x^2(1 + 2x/(2n+1))^{2n-1} = ((2n-1)/(2n+1))^{2n-1}.$$

If $x = \gamma_n$ is a solution, then as n is replaced by $n+1$, for fixed x, the right side decreases and the left side increases. Clearly x must decrease then to preserve equality and so $\gamma_{n+1} < \gamma_n$.

Remark 5. The special case $n = 1$ in Example 3 gives Maekawa's result [9].

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